Hypothesis Testing

Hypothesis test for binomial population proportion, p.

It is often of interest to test a particular value for a population proportion. Is the percentage of supporters for a particular referendum, 62%? Is the proportion of dermatologists who are women 28%?

One-sided Alternative Hypothesis with \( H_1: p < p_0 \)

We can use the following test if

\[
    n * p_0 > 5 \quad \text{and} \quad n * (1 - p_0) > 5, 
\]

where \( p_0 \) is the hypothesized proportion in the null hypothesis

1. \( H_0: p = p_0 \)
2. \( H_1: p < p_0 \)
3. Significance level is \( \alpha \).
4. The test statistic (T.S.) is \( z = \frac{\hat{p} - p_0}{\frac{p_0(1 - p_0)}{n}} \) where \( \hat{p} = \frac{x}{n} \).

5. p-value = \( P(z \leq \text{test. stat.value}) \)

OR

The critical value will be the negative of \( Z_{(\alpha)} \).

6. Decision Rule:
   - If using **p-value method** we would reject \( H_0 \) if p-value \( \leq \alpha \).
     Do not reject \( H_0 \) if p-value \( > \alpha \).
   - If using **Critical Value Method** we reject \( H_0 \) if T.S. \( \leq \) critical value
     Do not reject \( H_0 \) if T.S. \( < \) critical value

7. State conclusion (and conditions)

Example:

It has long been accepted that the proportion of people who live in New York City and attend at least one Broadway play each year is 14%. However, some in the community have suggested that this percentage has declined in the last 15 years. A study is commissioned to sample 1200 individuals from the five boroughs of NYC and to estimate the percentage of people in NYC that attended at least one play in the last year. 135 of
the 1200 people interviewed stated that they had attended at least one play in the last
year. Test the hypothesis that the percentage of people in NYC that attend at least one
play per year is less than 14% using $\alpha = 0.05$.

The sample proportion of successes is $\hat{p} = \frac{x}{n} = \frac{135}{1200} = 0.1125$

First check the conditions

$$n * p_0 = 1200 * 0.14 = 168 > 5$$

$$n * (1 - p_0) = 1200 * (1 - 0.14) = 1032 > 5$$

So we can use the following to test the hypothesis that $p = 0.14$ against the alternative
hypothesis $p < 0.14$.

1. $H_0$: $p = 0.14$
2. $H_1$: $p < 0.14$
3. Significance level is $\alpha=0.05$
4. The test statistics (T.S.) is

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 (1-p_0)}{n}}} = \frac{0.1125 - 0.14}{\sqrt{\frac{0.14(1-0.14)}{1200}}} = -2.745$$

5. $p$-value = $P(z \leq -2.74)$

Using Standard Normal Table we determine that $p$-value = 0.0031

The critical value will be the negative of $Z_{\alpha} = Z_{0.05} = 1.645$. So C.V. = -1.645

6. Decision Rule:

If using **p-value method** we would reject $H_0$ if $p$-value $\leq \alpha$.

Do not reject $H_0$ if $p$-value $> \alpha$.

Because 0.0031 $\leq 0.05$ we reject $H_0$

If using **Critical Value Method** we reject $H_0$ if T.S. $\leq$ critical value

Do not reject $H_0$ if T.S. $> \text{critical value}$

Because -2.745 $\leq -1.645$ we reject $H_0$

7. Conclusion: At the 5% significance level we conclude that the percentage of all New
York City residents who attended at least one play last year was significantly less than
14%.

**One-sided Alternative Hypothesis with $H_1$:** $p > p_0$

We can use the following test if
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\[ n^* p_0 > 5 \] and \[ n^*(1 - p_0) > 5, \]

where \( p_0 \) is the hypothesized proportion in the null hypothesis

1. \( H_0: p = p_0 \)
2. \( H_1: p > p_0 \)
3. Significance level is \( \alpha \).
4. The test statistic (T.S.) is \( z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} \) where \( \hat{p} = \frac{x}{n} \).
5. \( p \)-value = \( P(z \geq \text{test. stat.value}) \)

OR

The Critical value will be \( Z_{(\alpha)} \).

6. Decision Rule:

   If using **p-value method** we would reject \( H_0 \) if \( p \)-value \( \leq \alpha \).
   Do not reject \( H_0 \) if \( p \)-value >\( \alpha \).

   If using **critical value method** we reject \( H_0 \) if T.S. \( \geq \) critical value
   Do not reject \( H_0 \) if T.S. < critical value

7. State conclusion (and conditions)

**Example:**

Computer Company FGH claims that 3% of their manufactured CPUs are defective. A consumer advocate believes that this percent is higher than 3%. The consumer advocate obtains a random sample of 250 FGH CPUs and finds defects in 15 CPUs. Does the data support the claim of the consumer advocate? Use \( \alpha = 0.10 \).

The sample proportion of successes is \( \hat{p} = \frac{x}{n} = \frac{15}{250} = 0.06 \)

First check the conditions

\[ n^* p_0 = 250 \times 0.03 = 7.5 > 5 \]
\[ n^*(1 - p_0) = 250 \times (1 - 0.03) = 242.5 > 5 \]

So we can use the following to test the hypothesis that \( p = 0.03 \) against the alternative hypothesis \( p > 0.03 \).
1. \( H_0: \ p = 0.03 \)
2. \( H_1: \ p > 0.03 \)
3. Significance level is \( \alpha = 0.10 \)

4. The test statistic (T.S.) is
\[
    z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} = \frac{0.06 - 0.03}{\sqrt{0.03(1 - 0.03)/250}} = 2.7806
\]

5. \( p \)-value = \( P(z \geq 2.78) \)
   Using Standard Normal Table we determine that \( p \)-value = 0.0027

   The critical value will be \( Z_{(\alpha)} = Z_{(0.10)} = 1.28 \). So C.V. = 1.28

6. Decision Rule:
   - If using **p-value method** we would reject \( H_0 \) if \( p \)-value \( \leq \alpha \).
   - Do not reject \( H_0 \) if \( p \)-value \( > \alpha \).
   - Because 0.0027 \( \leq \) 0.10 we reject \( H_0 \)

   - If using **Critical Value Method** we reject \( H_0 \) if T.S. \( \geq \) critical value
   - Do not reject \( H_0 \) if T.S. < critical value
   - Because 2.745 \( \geq \) 1.28 we reject \( H_0 \)

7. Conclusion: At the 5 % significance level we conclude that the percentage of all FGH computer CPUs that are defective is significantly greater than 3%.

**Two-sided Alternative Hypothesis**

To perform a two-sided hypothesis test for the population proportion, we can use the following test if

\[
n * p_0 > 5 \quad \text{and} \quad n * (1 - p_0) > 5,
\]

where \( p_0 \) is the hypothesized proportion in the null hypothesis

1. \( H_0: \ p = p_0 \)
2. \( H_A: \ p \neq p_0 \)
3. Significance level is \( \alpha \).

4. The test statistic (T.S.) is
\[
    z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} \text{ where } \hat{p} = \frac{x}{n}.
\]

5. \( p \)-value = \( 2 \times P(z \geq |\text{T.S.}|) \)
   The critical value will be \( Z_{(\alpha/2)} \)
6. Decision Rule:
   If using **p-value method**, 
   We would reject $H_0$ if p-value $\leq \alpha$.
   Do not reject $H_0$ if p-value $> \alpha$.

   If using **Critical Value Method**, 
   We reject $H_0$ if $T.S. \leq -\text{critical value}$ or $T.S. \geq \text{critical value}$
   Otherwise do not reject $H_0$
   OR
   We reject $H_0$ if $|T.S.| \geq \text{critical value}$
   Otherwise do not reject $H_0$

7. State Conclusion (and conditions)

**Example:**
Suppose we think the population proportion is 82%. We obtain a random sample of 94 individuals and find that from our sample 75 of 94 meet the criterion. With $\alpha = 0.01$ can we claim that the population proportion is different from 82%.

The sample proportion of successes is $\hat{p} = \frac{x}{n} = \frac{75}{94} = 0.79787$

First check the conditions

\[
n * p_0 = 94 * 0.82 = 77.08 > 5
\]

\[
n * (1 - p_0) = 94 * (1 - 0.82) = 16.92 > 5
\]

So we can use the following to test the hypothesis that $p = 0.82$ against the alternative hypothesis $p \neq 0.82$.

1. $H_0: p = 0.82$
2. $H_1: p \neq 0.82$
3. Significance level is $\alpha = 0.01$
4. The test statistics is $z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{0.80 - 0.82}{\sqrt{\frac{0.82(1 - 0.82)}{94}}} = -0.02 \approx -0.5047$

   using rounded value for $\hat{p} = \frac{75}{94} = 0.80$.

5. p-value = $2 * P(z \geq |T.S.|) = 2 * P(z \geq \left| -0.5047 \right|) = 2 * P(z \geq 0.50) = 2 * 0.3085 = 0.6170$

   The critical value will be $Z_{(\alpha/2)} = Z_{(0.005)} = 2.578$
6. Decision Rule:
   If using **p-value method**, we would reject $H_0$ if $p$-value $\leq \alpha$.
   Do not reject $H_0$ if $p$-value $> \alpha$.
   We do not reject $H_0$ because $0.6170 > 0.05$.

   If using **Critical Value Method**, we reject $H_0$ if $T.S. \leq -$ critical value or $T.S. \geq$ critical value.
   Otherwise do not reject $H_0$.
   OR
   We reject $H_0$ if $|T.S.| \geq$ critical value.
   Otherwise do not reject $H_0$.
   We reject $H_0$ because $|T.S.| (0.5047) <$ critical value $(2.578)$.

7 Conclusion: At the 1% significance level we conclude that the population proportion’s value is not significantly different from 0.82.

**Example:**
The Association of Medical Actuaries estimates that 74% of all doctors are married. The Biomedical Bean-Counters Group feels this number is incorrect. They randomly select a sample of 120 doctors, contact them and determine whether or not they are married. From their sample of the 120 doctors, 95 were married. Using $\alpha = 0.10$, test whether or not the claim of 74% is still valid.

The sample proportion of successes is $\hat{p} = \frac{x}{n} = \frac{95}{120} = 0.791667$.

First check the conditions:

\[ n * p_0 = 120 * 0.74 = 88.8 > 5 \]
\[ n * (1 - p_0) = 120 * (1 - 0.74) = 31.2 > 5 \]

So we can use the following to test the hypothesis that $p = 0.74$ against the alternative hypothesis $p \neq 0.74$.

1. $H_0$: $p = 0.74$
2. $H_1$: $p \neq 0.74$
3. Significance level is $\alpha = 0.10$

4. The test statistics is $z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{0.792 - 0.74}{\sqrt{\frac{0.74(1 - 0.74)}{120}}} = \frac{0.052}{0.0400} = 1.2986$.
using rounded value for $\hat{p} = \frac{95}{120} = 0.792$.

5. p-value = $2* P(z \geq |T.S.|) = 2* P(z \geq |1.2986|) = 2*P(z \geq 1.30) = 2*0.0968 = 0.1936$

The critical value will be $Z_{\frac{\alpha}{2}} = Z_{(0.05)} = 1.645$

6. Decision Rule:
   - If using **p-value method,**
     - We would reject $H_0$ if p-value $\leq \alpha$.
     - Do not reject $H_0$ if p-value $> \alpha$.
   - We do not reject $H_0$ because $0.1936 > 0.10$.

   If using **Critical Value Method,**
   - We reject $H_0$ if $T.S. \leq$ - critical value or $T.S. \geq$ critical value
   - Otherwise do not reject $H_0$
   - Since the T.S. (1.2986) $<$ critical value (1.645) and T.S. (1.2986) $>$ - critical value (-1.645),
     - we do not reject the null hypothesis.
   - OR
     - We reject $H_0$ if $|T.S.| \geq$ critical value
     - Otherwise do not reject $H_0$
   - We do not reject $H_0$ because $|T.S.|$ (1.2986) $<$ critical value (1.645)

7. Conclusion: At the 10% significance level we conclude that the proportion of all doctors who are married is not significantly different from 0.74.