RECENT TRENDS AND APPROACHES

TO THE ANALYSIS OF

THE SURPLUS PROCESS

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INTRODUCTION

- classical Poisson model to describe risk characteristics of portfolio of insurance business

- number of claims process \( \{N_t; t \geq 0\} \) is a Poisson process with rate \( \lambda \)

- claim sizes \( \{Y_1, Y_2, \cdots\} \) are iid with common df
  \[ H(y) = 1 - \overline{H}(y) = Pr(Y_i \leq y) \]
  \[ E(Y) = \int_0^\infty ydH(y) < \infty \]

- total or aggregate claims process \( \{S_t; t \geq 0\} \) where
  \[ S_t = \sum_{i=1}^{N_t} Y_i \]

- premiums paid continuously at rate \( c \) per unit time where
  \[ c = \lambda E(Y)(1 + \theta) \]

and \( \theta > 0 \) is the relative security leading
- risk characteristics of insurance portfolio analyzed via surplus process \( \{U_t; t \geq 0\} \) where

\[
U_t = u + ct - S_t
\]

- very simple model

- danger and volatility of portfolio

- parameters under control of insurer

  - initial surplus \( u = U_0 \)

  - loading \( \theta \)

- time of ruin \( T = \inf\{t; U_t < 0\} \) with \( T = \infty \) if \( U_t \geq 0 \) for \( t \geq 0 \)

  - historical analysis focussed on ruin probability

\[
\psi(u) = Pr(T < \infty) = E\{I(T < \infty)\}
\]
- earliest results are approximate in nature

- Cramer-Lundberg asymptotic formula

\[ \psi(u) \sim Ce^{-\kappa u}, u \to \infty \]

- Lundberg’s inequality

\[ \psi(u) \leq e^{-\kappa u}, u \to \infty \]

where \( \kappa > 0 \) is the adjustment coefficient satisfying

\[ 1 + \kappa(1 + \theta)E(Y) = E(e^{\kappa Y}) \]

- exact compound geometric tail representation

\[ \psi(u) = \sum_{n=1}^{\infty} \{1 - \psi(0)\} \{\psi(0)\}^n H_1^*(u), u \geq 0 \]

where \( \psi(0) = \frac{1}{1+\theta} \) and

\[ H_1(u) = \int_0^u H(y)dy/E(Y) \]

is the ladder height, equilibrium, or integrated tail df
- explicit evaluation by summation or analytic Laplace transform inversion was traditional approach

- traditional approximations
  - Beekman Bowers gamma approximation
  - defective renewal equation approaches

- alternative series expansions have proved to be useful recently (e.g. Gerber, Shiu, Picard, Lefevre)

- numerical approaches
  - Panjer type recursions
  - recent advances in numerical inversion of Laplace transforms (by queueing theorists)
- $\psi(u)$ is also of much interest in the well known $M/G/1$ queue discipline
  - tail of limiting or equilibrium actual waiting time distribution
    - $1/(1 + \theta)$ is utilization factor
    - service df is $H(y)$
  - heavy traffic approximations
    - classical exponential
    - Tijm’s two term exponential
    - Choudhury, Lucantoni, and Whitt’s generalization
- phase type distributions
  - time to absorption in continuous time Markov process
  - includes most common distributions with rational Laplace transform (any many others)
- analytic advances
  - unifies and extends approaches to evaluation of $\psi(u)$ by analytic Laplace transform inversion
- numerical advances
  - well-suited for exact numerical evaluation of $\psi(u)$
  - requires evaluation of matrix-exponentials
GENERALIZATIONS OF THE CLASSICAL POISSON MODEL

- much recent work on models with more general premium structures (deterministic and stochastic) as well as models incorporating interest or investment income, dividends via barrier strategy, dependency between claim numbers and amounts

- less mathematical tractability in general

- approximations/explicit results often available when aggregate claims process follows a diffusion or a compound Poisson process plus a diffusion

- tractable generalizations involve discrete time and state space models, as well as models where Poisson arrival rate, claim sizes, and premium rate are state dependent

  - Markov modulated models

- simple generalization is when number of claims process

\[ \{N_t; t \geq 0\} \text{ is a renewal process} \]
RENEWAL RISK MODEL

- often referred to as Sparre Andersen model

- time between claims has df $K(t) = 1 - \bar{K}(t)$ instead of exponential special case $K(t) = 1 - e^{-\lambda t}$

- by Wiener-Hopf techniques, compound geometric tail formula holds, i.e.

$$
\psi(u) = \sum_{n=1}^{\infty} \{1 - \psi(0)\}^{n} F^{*n}(u)
$$

- probabilistic interpretation

- many properties follow from this representation, including the Cramer-Lundberg asymptotic formula and Lundberg’s inequality

- $\psi(u)$ is the tail of limiting or equilibrium actual waiting time distribution in G/G/1 queue

- $\psi(0)$ and $F(u) = 1 - \bar{F}(u)$ have only been identified for some choices of $H(y)$ and/or $K(t)$
- evaluation of $\psi(u)$ possible if either $H(y)$ or $K(t)$
- is of phase type
  - by identifying $\psi(0)$ and $F(u)$
- has a rational Laplace transform
  - by identifying the Laplace transform
  - Tijm’s approximation is useful here
- stationary renewal risk model
  - same as renewal risk model, except time until first claim has pdf $\frac{K(t)}{\int_0^\infty K(v)dv}$
  - also generalization of classical Poisson model, perhaps more realistic
- ruin probability $\psi_s(u)$ is the tail of compound geometric convolution, i.e. $\psi_s(u) = Pr(L_s > 0)$ satisfies
  $$\psi_s(u) = \frac{1}{1 + \theta} \int_0^u \psi(u - y)dH_1(y) + \frac{1}{1 + \theta}H_1(u)$$
  - may be interpreted probabilistically
  - $\psi_s(u)$ is the tail of the limiting or equilibrium virtual waiting time distribution in the G/G/1 queue
OTHER RUIN THEORETIC FUNCTIONALS

- severity or deficit at ruin is the negative surplus $|U_T|$

- risk measure interpretation

  - initial surplus to fix probability of ruin essentially incorporates “Value at Risk” (VaR) concepts

  - mean deficit at ruin is of interest in connection with “Conditional Tail Expectation” (CTE) or “Tail VaR”

  - useful as financial risk management tool
let $\psi(u) = Pr(L > u)$ in renewal risk model, where $L$ is the maximum aggregate loss; and $V_u = |U_T| |(T < \infty)$ be an ‘independent’ deficit random variable (of $L$)

- $Pr(L > u + y|L > u) = Pr(L + V_u > y)$

- characteristics of deficit obtainable by deconvolution

- mean deficit is

$$E(V_u) = \int_0^\infty \left\{ \frac{\psi(u + y)}{\psi(u)} - \psi(y) \right\} dy$$

- essentially a property of compound geometric random variables; many (reliability) properties obtainable using this result ⇒ ruin theoretic results applicable elsewhere
- let $G_u(y) = 1 - \mathbb{G}_u(y) = Pr(V_u \leq y)$, and

$$G_u(y) = \frac{\int_0^u F_{u-t}(y) F(u-t) dPr(L \leq t)}{\int_0^u F(u-t) dPr(L \leq t)}$$

where $F_x(y) = 1 - F(x+y)/F(x)$

- mixture representation

- $G_u(y)$ is of phase type if $H(y)$ is of phase type; special cases also (mixtures or combinations of exponentials or Erlangs) due to phase type parameter invariance

- simpler as $u \to \infty$

- other analytic properties follow from mixture

- analogous convolution result holds in stationary renewal risk model as well, i.e.

$$Pr(L_s > u+y | L_s > u) = Pr(L + V^s_u > y)$$

where $V^s_u$ is independent of $L$

- essentially a property of compound geometric convolutions
GERBER-SHIU DISCOUNTED PENALTY FUNCTION

- originally motivated in part by option pricing interpretation

- the surplus immediately before ruin is $U_{T-}$

- let $\delta > 0$ be a discount factor, and $w(x_1, x_2)$ a nonnegative function ($x_1$ is a surplus variable, and $x_2$ a deficit variable)

- let

$$m(u) = E\{e^{-\delta T}w(U_{T-}, |U_T|)I(T < \infty)\}$$

be the Gerber-Shiu “discounted penalty” function

- special cases

  1) $\delta = 0$ and $w(x_1, x_2) = 1 \Rightarrow m(u) = \psi(u)$

  2) $w(x_1, x_2) = 1 \Rightarrow m(u) = E\{e^{-\delta T}I(T < \infty)\}$ is the Laplace transform of the defective distribution of the time to ruin

  3) $\delta = 0$ and $w(x_1, x_2) = I(x_2 \leq y) \Rightarrow m(u) = G(u, y)$ is the defective df of the deficit

  4) surplus df, moments, etc
- in classical model, \( m(u) \) satisfies a defective renewal equation

- \( \rho = \rho(\delta) \) unique nonnegative solution of

\[
c\rho + \lambda \int_0^\infty e^{-\rho y} dH(y) = \lambda + \delta
\]

- let

\[
\phi = \frac{\int_0^\infty e^{-\rho y} dH_1(y)}{1 + \theta}
\]

- let the df \( B(x) = 1 - \overline{B}(x) \) satisfy

\[
\overline{B}(x) = \frac{\int_0^\infty e^{-\rho t} \overline{H}(x + t) dt}{\int_0^\infty e^{-\rho t} \overline{H}(t) dt}
\]

- then

\[
m(u) = \phi \int_0^u m(u - t) dB(t) + e^{\rho u} \int_u^\infty e^{-\rho t} \tau(t) dt
\]

where

\[
\tau(t) = \frac{1}{(1 + \theta) E(Y)} \int_t^\infty w(t, y - t) dH(y)
\]
- time of ruin

- $w = 1 \Rightarrow$ Laplace transform $m(u) = E \{ e^{-\delta T} I(T < \infty) \}$ satisfies

$$m(u) = \phi \int_0^u m(u - t)dB(t) + \phi B(u)$$

- compound geometric tail solution

$$m(u) = \sum_{n=1}^{\infty} (1 - \phi) \phi^n B^n(u)$$

- may be inverted to get defective density in exponential case $H(y) = 1 - e^{-y/E(Y)}$

- let $\alpha_k(u) = E \{ T^k I(T < \infty) \}, k = 0, 1, 2, \cdots$, and differentiation yields the defective renewal difference equations

$$\alpha_k(u) = \frac{1}{1 + \theta} \int_0^u \alpha_k(u - y)dH_1(y) + \frac{k}{c} \int_u^\infty \alpha_{k-1}(y)dy$$

- $\alpha_0(u) = \psi(u)$

- can solve to get moments (complicated)

- can also get moments of (discounted) deficit, joint moments of time and deficit, etc.
SUMMARY

- more in-depth analysis of classical Poisson and more general models
  
  - mathematical and numerical advances
  
  - related quantities of interest
    
    - deficit at ruin $|U_T|$
    
    - surplus immediately prior to ruin $U_T^-$
    
    - claim causing ruin $|U_T| + U_T^-$
    
    - duration of negative surplus
    
    - maximum severity of ruin
    
    - maximum surplus prior to ruin
  
  - interdisciplinary
    
    - queueing theory and applied probability
    
    - risk management