In research, we deal with sample data taken from a population. Due to limitations of resources (time, equipment, etc) and the destructive nature of certain experiments, we cannot collect data from every element in the population.

Sample data are used to make general statements (inference) about the whole population. The branch of statistics that deals with making conclusions or inference about a population based on sample data is called STATISTICAL INFERENC.

* When we make inference or conclusion about a population based on sample data, we are not absolutely certain about our conclusion. Why?

This is because, the sample data do not contain all the information in the population. The sample only contains partial information.

* What is the only way to become absolutely certain about our conclusion?

If we collect data from every element (unit) in the population.

Why do we study probability? Probability is a tool that we use to measure such uncertainty.

But we are not absolutely certain about the inference since the sample contains partial information.
1) An Experiment: is any action or process that generates observations.

   Example:
   - Tossing a coin once (or several times)
   - Drawing a card from a deck
   - Measuring weight of students
   - Sampling customers for an opinion survey

2) A Sample Space is a set consisting all possible outcomes of an experiment. It is denoted by \( S \).

   Example:
   a) Tossing a coin once \( S = \{ H, T \} \)
   b) Tossing a coin twice \( S = \{ HH, HT, TH, TT \} \)
   c) Tossing a die once \( S = \{ 1, 2, 3, 4, 5, 6 \} \)

Each outcome in a sample space is called an elementary outcome or a simple event.

Example: In example (b) above there are four simple events.

3) An event is a subset of a sample space, denoted by any capital letter.

Example: Suppose the experiment is tossing a coin twice

   Let A be the event of getting no heads
   Let B be the event of getting at least one head

Find the outcomes contained in events A \( \cap \) B

   \[ A = \{ TT \} \]
   \[ B = \{ HH, HT, TH \} \]
An Event

A Simple event contains exactly one elementary outcome

ex: \( A = \{ TT \} \)

A compound event contains two or more outcomes.

ex: \( B = \{ HH, HT, TH \} \)

Note: Each time an expr is performed one and only one elementary outcome can occur. We say that an event \( A \) has occurred when any of the elementary outcome in \( A \) occurs.

Example: Roll a die once

let \( A \) be the event that the resulting outcome is \( \leq 3 \)
let \( B \) " " " " " " " " " " is odd
let \( C \) " " " " " " " " " " is even

\[
A = \{ 1, 2, 3 \}
\]

\[
B = \{ 1, 3, 5 \}
\]

\[
C = \{ 2, 4, 6 \}
\]

Suppose after our expr, the resulting outcome is 3. Which of the above events has occurred? \( A \) and \( B \)

Probability and Its Properties

Probability of an event: It is a numerical value that indicates how likely it is that the event will occur. It is the proportion of times the event is expected to occur in many repeated trials of the expr.

If \( A \) is an event, \( P(A) \) is probability of event \( A \).
Properties of Probability

1. Probability of an event is always between 0 and 1.
   \[ 0 \leq P(A) \leq 1 \]

2. Probability of an event is the sum of the probabilities assigned to all the elementary outcomes contained in the event.
   \[ P(A) = \sum \text{prob. of all the outcomes in event } A \]

Example: Toss a coin twice. Let \( A \) be the event of getting exactly one head.
   \[ A = \{HT, TH\} \]
   \[ P(A) = P\{HT\} + P\{TH\} \]

3. The sum of probabilities of all the elements of \( S \) must be 1.
   \[ P(S) = 1 \]

\( S \) = denotes sample space

Is \( S \) an event?

Note for an event \( A \)

\[ P(A) = 0 \Rightarrow \text{means Event } A \text{ can not occur} \]
\( (A \text{ is an impossible event}) \)

\[ P(A) \text{ close to zero} \Rightarrow \text{means } A \text{ is a rare event} \]
\( (A \text{ is an rare event}) \)

\[ P(A) = 1 \Rightarrow \text{means Event } A \text{ must always occur} \]
\( (A \text{ is a sure event}) \)
1. "Theoretical" or "a priori" definition of Probability

If each trial of an exp't results in *n* equally likely outcomes and *r* of these outcomes belong to an event *E*. Then prob. of event *E* is

\[ P(E) = \frac{r}{n} = \frac{\text{number of outcomes in event } E}{\text{number of outcomes in sample space}} \]

If all the outcomes of an exp't are equally likely, then we can calculate prob. of an event without actually performing the experiment.

This is why we call it "Theoretical" or "a priori" probability of an event.

**Example (a)** rolling a perfect die and recording the face.

\[ S = \{1, 2, 3, 4, 5, 6\} \quad n = 6 \text{ equally likely outcomes} \]

Let A be that the outcome is < 4

\[ A = \{1, 2, 3\} \quad r = 3 \]

\[ P(A) = \frac{r}{n} = \frac{3}{6} = \frac{1}{2} \]

(b) a bag contains 3 red and 7 white balls. We randomly select a ball from the bag.

\[ A = \text{the chosen ball is white} \]

\[ n = 10 \quad r = 7 \quad P(A) = \frac{r}{n} = \frac{7}{10} = 0.7 \]
2. Relative frequency definition of probability

When the elementary outcomes of an exp are not equally likely, how do we assign or estimate prob of an event?

by repeating the exp a large number of times and recording the number of times an event occurs.

Let \( n \) be the number of repetitions or trials of an exp.
Let \( A \) be the event and \( r \) be the \( n \) trials result in event \( A \)

Then

\[
\text{The relative frequency} = \frac{r}{n} \rightarrow \text{this is called the empirical prob of an event}
\]

\[
\downarrow
\]

\[
\text{used to estimate } p(A)
\]

Note

a) \( n \) must be large. The larger the number of trials \( n \), the better is the estimate of the probability.

b) The trials must be independent - means the outcome of any of the trials does not depend on the results of the previous trials.

c) as \( n \) increases, relative frequency of event \( A \) approaches the theoretical probability \( P(A) \).

Example: roll a die 100 times

\( A \) = the outcome is 6

Suppose six occurs 23 times

\[
p(A) = \frac{r}{n} = \frac{23}{100} = 0.23\]
Event Relations

1. Complement
2. Union
3. Intersection

The complement of an event $A$: is the set containing all outcomes that are not in $A$; denoted by $A'$.

The shaded portion is $A'$.

Note: If $A$ occurs, then $A'$ will not occur.

The union of two events $A$ and $B$: is the set containing outcomes that are either in $A$ or $B$ or both. It is denoted by $A \cup B$.

The intersection of two events $A$ and $B$: is the set containing outcomes that are in both $A$ and $B$; denoted by $A \cap B$. 
Mutually exclusive events: Two events $A$ and $B$ are mutually exclusive (disjoint) if their intersection is an empty set. i.e.,

$$A \cap B = \emptyset$$

Means $A$ and $B$ do not have common outcomes.
$\Rightarrow$ The occurrence of $A$ excludes $B$ and vice versa.
$\Rightarrow$ The two events cannot occur together.
$\Rightarrow P(A \cup B) = 0$

Laws of Probability

1. Law of Complementation
2. Law of Addition ("Or" Law)
3. Law of Multiplication ("And" Law)
1) Law of complementation
   \[ P(A) = 1 - P(A') \quad \text{or} \quad P(A') = 1 - P(A) \]

2) Law of addition
   \[ P(A \text{ or } B \text{ or both}) = P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
   "If A and B are mutually exclusive, then
   \[ P(A \cup B) = P(A) + P(B) \quad \text{why?} \]

3) Multiplication Law
   \[ P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B) \]

\underline{Conditional Probability}

The conditional prob. of A given B is denoted by \( P(A|B) \) and is defined as
\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

Similarly, \[ P(B|A) = \frac{P(A \cap B)}{P(A)} \]
Two events A and B are independent if
\[ P(A|B) = P(A) \]

Similarly, \[ P(B|A) = P(B) \]
or \[ P(\text{A and B}) = P(A) \cdot P(B) \]

If A and B are independent, it means that knowing the fact that B has occurred has no effect on occurrence of event A, i.e.
\[ P(A|B) = P(A) \]

Example: roll a fair die once

A = the outcome is even
B = the outcome is < 5

\[ A = \{2, 4, 6\} \quad B = \{1, 2, 3, 4\} \]
\[ P(A) = \frac{3}{6} = \frac{1}{2} \quad P(B) = \frac{4}{6} = \frac{2}{3} \]

\[ A \cap B = \{2, 4\} \quad P(A \cap B) = \frac{2}{6} \]
\[ P(A \cup B) = \frac{5}{6} \]

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
\[ = \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{3 + 4 - 2}{6} = \frac{5}{6} \]
Below are the types and ages of 4 monkeys:

<table>
<thead>
<tr>
<th>Monkey</th>
<th>Type</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Baboon</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>Baboon</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>Spider</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>Spider</td>
<td>6</td>
</tr>
</tbody>
</table>

The events are randomly selecting two monkeys:

- $A$: the monkeys are the same type
- $B$: the monkeys are the same age
- $C$: the monkeys are different types

Find the sample space, elements in $A$, $B$, $C$, $A'$, $A \cup B$, $A \cap B$, $B \cap C$

$S = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}$

$A = \{ (1,2), (3,4) \}$

$B = \{ (1,3), (1,4), (3,4) \}$

$C = \{ (1,3), (1,4), (2,3), (2,4) \}$

$A' = \{ (1,3), (1,4), (2,3), (2,4) \}$

$A \cup B = \{ (1,2), (3,4), (1,3), (1,4) \}$

$A \cap B = \{ (3,4) \}$

$B \cap C = \{ (1,3), (1,4) \}$
Example on Law of Complement: \( P(\text{at least one}) = 1 - P(\text{none}) \)

(a) Randomly select one card from a deck of 52

- \( A \): the card is an ace
- \( A' \): the card is not an ace

\[
P(A') = 1 - P(A) = 1 - \frac{4}{52} = \frac{48}{52}
\]

\[
P(A) + P(A') = 1
\]

(b) A child is given three word association problems. For each problem, there are two suggested answers, one is correct and the other is wrong. The child has no understanding of the words and answers by guessing. What is the probability of getting at least one correct answer?

Solution:

- 1st question: \( C \)
- 2nd question: \( W \)
- 3rd question: \( C \)

\( S = \{ \text{ccc, ccw, cwc, cww, wcc, wcw, wwc, www} \} \)

- \( A \): at least one correct answer
- \( A' \): no correct answer

\[
\begin{align*}
\text{\( n = 8 \) equally likely outcomes.} \\
\text{\( P(A') = \frac{1}{n} = \frac{1}{8} \)} \\
\text{\( P(A') = 1 - \frac{1}{8} = \frac{7}{8} \)} \\
\text{\( P(A) = 1 - P(A') = 1 - \frac{7}{8} = \frac{1}{8} \)}
\end{align*}
\]
Example on addition law

Roll a die once

\[ A = \text{outcome is even} \quad A = \{2, 4, 6\} \]
\[ B = \text{outcome is } < 5 \quad B = \{1, 2, 3, 4\} \]

\[ A \cap B = \{2, 4\} \]

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
\[ = \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6} \]

prob. that the outcome is either even or < 5 or both is \(\frac{5}{6}\).

Example on multiplication law

EX: A box contains 14 white and 6 red marbles. Randomly select two marbles without replacement.

Find prob. both marbles are red.

\[ A = \text{1st is red} \]
\[ B = \text{2nd is red} \]

\[ P(A \cap B) = P(A) \cdot P(B|A) = \frac{6}{20} \cdot \frac{5}{19} = 0.08 \]

EX:
From 25 students, 20 have checking account and 5 do not have a checking account. Randomly select two students with replacement. What is the prob. that both have no checking account?
Example on conditional probability

<table>
<thead>
<tr>
<th></th>
<th>Female</th>
<th>Male</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior</td>
<td>10</td>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>Junior</td>
<td>20</td>
<td>15</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>20</td>
<td>50</td>
</tr>
</tbody>
</table>

The exper is randomly select one student.
What is the prob. that
a) the student is female $P(F) = \frac{30}{50}$

b) the student is junior given that it is female
$P(J/F) = \frac{P(J \cap F)}{P(F)} = \frac{20}{50} = \frac{20}{30}$

c) the student is female given that it is senior
$P(F/S) = \frac{P(F \cap S)}{P(S)} = \frac{10}{50} = \frac{10}{15}$

Disjoint Events

Note: If two events are disjoint then


$$P(A \cap B) = 0$$
$$P(A/B) = 0$$
$$P(B/A) = 0$$

\[ P(A \cup B) = P(A) + P(B) \quad \text{b/c} \quad P(A \cap B) = 0 \]

Example on independent

roll a die

$A =$ odd outcome $A = \{ 1,3,5 \}$

$B =$ outcome $\geq 3 \quad B = \{ 3,4,5,6 \}$

$A \cap B = \{ 3,5 \}$

Are $A$ and $B$ independent?
If they are, then $P(A \cap B) = P(A) \cdot P(B) \quad 0 = \frac{3}{6} \cdot \frac{3}{6}$

$A$ and $B$ are mutually exclusive.
Mutually exclusive and exhaustive events

Two events $A$ and $B$ are said to be mutually exclusive and exhaustive if

$p(\text{A} \cap \text{B}) = 0$ and
$p(\text{A} \cup \text{B}) = 1$

Ex. roll a die

$A = \text{outcome even}$ $\quad A = \{2, 4, 6\}$
$B = \text{outcome odd}$ $\quad B = \{1, 3, 5\}$

$p(\text{A} \cap \text{B}) = 0$
$p(\text{A} \cup \text{B}) = p(\text{S}) = 1$

Odds of an event $A$ = $\frac{p(\text{A})}{1-p(\text{A})}$ = $\frac{p(\text{A})}{p(\text{A}')}$

If $p(\text{A}) = \frac{3}{4}$

Odds in favor of event $A$ = $\frac{3/4}{1-3/4} = \frac{3/4}{1/4} = \frac{3}{1}$

= 3 to 1
Summary (Chapter 5)

1. General Addition Law (for any events $A$ and $B$)
   \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

2. Special Addition Law (for disjoint events $A$ and $B$)
   \[ P(A \cup B) = P(A) + P(B) \]

3. General Multiplication Law (for any events $A$ and $B$)
   \[ P(A \cap B) = P(A) \cdot P(B/A) \]
   \[ = P(B) \cdot P(A/B) \]

4. Special Multiplication Law (for independent events $A$ and $B$)
   \[ P(A \cap B) = P(A) \cdot P(B) \]

5. Law of Complement
   \[ P(A) = 1 - P(A^c) \]

6. Checking independence of two events $A$ and $B$
   Two events are independent if
   \[ P(A/B) = P(A) \]
   \[ P(B/A) = P(B) \]
   \[ P(A \cap B) = P(A) \cdot P(B) \]

7. Probability, percentage, and odds
   Example: if $P(A) = \frac{3}{4}$
   \[ = 0.75 \]
   = 75%