

Chapter 9
Confidence Interval Estimation

Statistical inference - deals with drawing conclusions about a popn parameter(s) based on analysis of sample data. This will be the topic for the remainder of the course.

Types of Statistical Inference

1. Estimation (Chapter 9)
   a. Point estimation
   b. Confidence Interval estimation

2. Hypothesis testing (Chapter 10)

Point estimation: The objective of point estimation is to calculate (obtain), from a sample data, a single number that best approximate the popn parameter.

Popn with unknown parameters

\[ \mu \]

\[ S^2 \]

Sample data

\[ \bar{x} \]

\[ s^2 \]

\( \bar{x} \) is a point estimator of \( \mu \) \( \Rightarrow \) our best guess of the value of \( \mu \) is \( \bar{x} \).

\( s^2 \) is a point estimator of \( \sigma^2 \).

Note: A statistic used for estimating a parameter is called a point estimator and the standard deviation of an estimator is called its standard error (S.E.).

\( \bar{x} \) is a point estimator of \( \mu \) and the standard deviation of \( \bar{x} \) is referred to as standard error.

Properties of \( \bar{x} \):

a) \( E(\bar{x}) = \mu_{\bar{x}} = \mu \)

b) \( \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \text{standard error of } \bar{x} \)

c) When \( n > 30 \), \( \bar{x} \approx N \left( \mu_x, \frac{\sigma}{\sqrt{n}} \right) \)
Example 1: Suppose you want to assess the rate of employment in a given county. You take a random sample of 500 people and in this sample, 41 are unemployed. What is our estimate of the rate of unemployment (p)?

\[ \hat{p} = \frac{41}{500} = 0.082 \]
\[ = 8.2\% \text{ is our best guess of } p \]

Based on our sample, we estimate the true proportion of unemployed people (p) to be 8.2%. It is very important to understand that, if you take a different sample of people, your estimate of p changes.

Example 2: You want to determine the average height of female students at WVU. You take a sample of 32 students.

\[ \bar{x} = 163.4 \text{ cm} \]
\[ \bar{x} \text{ is our best guess of } \mu, \text{ so } \hat{\mu} = \bar{x} = 163.4 \text{ cm} \]

Disadvantage of point estimators: We don't know the accuracy of point estimates since it is based on a sample, the value of a point estimator varies from sample to sample. In some cases, it is larger than the true value of the parameter, and in others, it is smaller than the value of the parameter.

In general, a point estimator is subject to error because it is based on part of the entire collection of measurements in the population. Therefore, we want to know its margin of error.

To accomplish this, we need to include a statement of precision (how precise is the point estimate) by adding an error term as follows:

point estimate \pm error term

Example: In case of Example 2 above \( \Rightarrow 163.4 \pm 1 \text{ cm} \)
\[ [162.4 \text{ cm}, 164.4 \text{ cm}] \]
Estimate of the form \([162.4, 164.4]\) is called an interval estimate for \(\mu\) and all values in that interval are equally good estimators of \(\mu\).

Computing confidence interval for population mean, \(\mu\):

\[
\bar{x} \pm Z_{(1-\alpha)/2} \times \frac{s}{\sqrt{n}}
\]

Case I: When \(n\) is large \((n \geq 30)\)

Definition: A \((1-\alpha) \times 100\%\) confidence interval (CI) for \(\mu\) is given by

\[
\bar{x} \pm Z_{(1-\alpha)/2} \times \frac{s}{\sqrt{n}}
\]

Where

- \(\alpha\) = level of significance \((0 < \alpha < 1)\)
- \(1-\alpha\) = The confidence coefficient
- \((1-\alpha) \times 100\%\) = The level of confidence
- \(Z_{(1-\alpha)/2}\) = is the value of \(Z\) to the left of which the area is \(1-\alpha\).
- \(Z_{(1-\alpha)/2}\) = or the value of \(Z\) to the right of which the area is \(\alpha/2\).

\[
\text{area of shaded region} = 1-\frac{\alpha}{2}
\]

\[
\text{this area} = \frac{\alpha}{2}
\]

\[
S = \text{is the sample standard deviation}
\]

\[
\frac{Z_{(1-\alpha)/2} \times s}{\sqrt{n}} = \text{is the error term (margin of error)}
\]

\[
2 \left[ Z_{(1-\alpha)/2} \times \frac{s}{\sqrt{n}} \right] = \text{is the width of the confidence interval}
\]
The upper limit of the CI = UL = $\bar{x} + Z_{(1-\alpha/2)} \times \frac{s}{\sqrt{n}}$

The lower limit of the CI = LL = $\bar{x} - Z_{(1-\alpha/2)} \times \frac{s}{\sqrt{n}}$

Width of the CI = UL - LL = 2 (error term)

Example

Find a 95% confidence interval for the population mean height, $\mu$, of all female WVU students. To accomplish this task, a random sample of 32 female students were taken from the popn. Based on this sample, the average height and standard deviations were

$\bar{x} = 163.4 \text{ cm}$

$s = 6.1 \text{ cm}$

Solution

$n = 32 \Rightarrow N_{32}$

$\text{The popn}$

Heights of all female WVU students

$\mu =$ average height of all female WVU students

$\sigma =$ unknown parameter?

Heights of 32 female WVU students

The sample

$\bar{x} = 163.4 \text{ cm}$

$s = 6.1 \text{ cm}$

$\Rightarrow$ average height of 32 students

$\Rightarrow$ the SD of heights of 32 students

$\bar{x} \pm Z_{(1-\alpha/2)} \times \frac{s}{\sqrt{n}}$

$\Rightarrow (1-\alpha) \times 100\% = 95\%$

$1-\alpha = 0.95 \Rightarrow \alpha = 0.05$

$\Rightarrow Z_{(1-\alpha/2)} = Z_{(1-0.05/2)} = Z_{(1-0.025)} = Z_{0.975}$

$\bar{x} \pm Z_{(1-\alpha/2)} \times \frac{s}{\sqrt{n}} \Rightarrow 163.4 \pm 1.96 \times \frac{6.1}{\sqrt{32}}$

$\Rightarrow 163.4 \pm 1.96 \times 1.08$

$\Rightarrow 163.4 \pm 2.01$

$\Rightarrow [161.3, 165.5] \leftarrow \text{This is the 95% CI for } \mu$

The lower limit of the interval

The upper limit of the interval
We have now found the 95% CI for \( \mu \) = [161.3, 165.5], but how do we interpret this interval? (very important)

**Interpretation:** If we do the experiment a large number of times and calculate a 95% CI for each case (sample), then 95% of these calculated intervals will contain \( \mu \) and about 5% of the CIs will not contain \( \mu \).

Let us suppose that we are going to do the experiment 1000 times. It means that we are going to take samples of 32 observations 1000 times and calculate 1000 CIs.

<table>
<thead>
<tr>
<th>Sample #</th>
<th>( n )</th>
<th>Sample statistic</th>
<th>95% CI for ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32</td>
<td>( \bar{X}_1, s_1 )</td>
<td>( \bar{X}<em>1 \pm Z</em>{(1-\alpha/2)} \frac{s_1}{\sqrt{n}} \Rightarrow [UL_1, UL_1] )</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>( \bar{X}_2, s_2 )</td>
<td>( \bar{X}<em>2 \pm Z</em>{(1-\alpha/2)} \frac{s_2}{\sqrt{n}} \Rightarrow [UL_2, UL_2] )</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
<td>( \bar{X}_3, s_3 )</td>
<td>( \bar{X}<em>3 \pm Z</em>{(1-\alpha/2)} \frac{s_3}{\sqrt{n}} \Rightarrow [UL_3, UL_3] )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1000</td>
<td>32</td>
<td>( \bar{X}<em>{1000}, s</em>{1000} )</td>
<td>( \bar{X}<em>{1000} \pm Z</em>{(1-\alpha/2)} \frac{s_{1000}}{\sqrt{n}} \Rightarrow [UL_{1000}, UL_{1000}] )</td>
</tr>
</tbody>
</table>

So, we have 1000 confidence intervals for \( \mu \), \( \Rightarrow \) The theory says 95% of these intervals \( \Rightarrow \) contain the parameter \( \mu \), and 5% of these intervals \( \Rightarrow \) do not contain \( \mu \).

So if we go back to our example, the 95% CI for \( \mu \) was

\[ [161.3 \text{ cm}, 165.5 \text{ cm}] \]

This interval could be one of those intervals that contain \( \mu \).

\( \Rightarrow \) we are 95% confident about this

or This interval could be one of those intervals that do not contain \( \mu \).

\( \Rightarrow \) we are 5% confident about this.
Summary

Based on a single sample of n=32 observations, the 95% CI for μ was [161.3, 165.5].

Correct interpretation: We are 95% confident that the true mean height (μ) of female WVU students lies in the interval [161.3, 165.5] cm.

Wrong interpretation: \[ P(161.3 < μ < 165.5) = 0.95 \]

Do we certainly know that the value of μ lies between 161.3 and 165.5 cm?

No, μ may or may not be within the interval 161.3 to 165.5 cm.

Please read more on interpretation of a 95% CI — page 116 to 117 section 9.2 in your text.

Factors that affect the width of a confidence interval

1. The sample size (n)
2. The amount of variability (s) (standard deviation)
3. The level of confidence

Effect on width:

As n ↑ \Rightarrow we get a narrower confidence interval

Note: width of a CI = 2 (error term) = 2 \left( \frac{Z_{1-\alpha/2} \cdot s}{\sqrt{n}} \right)

So as n ↑ \Rightarrow error term becomes smaller

\Rightarrow width becomes smaller or narrower

\Rightarrow narrower width means greater precision

A small n \Rightarrow results in a wider confidence interval.
(3) Effect of \( S \) on width

\[
\text{width} = 2 \left( Z_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \right)
\]

As \( S \uparrow \rightarrow \) error term increases
\( \rightarrow \) wider CI
\( \Delta \) means smaller precision

(3) Effect of confidence level on width

Note that when
\[
\begin{align*}
\alpha \uparrow & \Rightarrow \text{level of confidence} \left[ (1-\alpha) \times 100\% \right] \downarrow \\
\alpha \downarrow & \Rightarrow \text{level of confidence} \left[ (1-\alpha) \times 100\% \right] \uparrow
\end{align*}
\]

**Example**

\( \alpha = 0.05 \Rightarrow \text{level of confidence} = (1-0.05) \times 100\% = 95\% \Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96 \)

\( \alpha = 0.01 \Rightarrow \text{level of confidence} = (1-0.01) \times 100\% = 99\% \Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{0.995} = 2.58 \)

This example illustrates that as \( \alpha \uparrow \Rightarrow Z_{1-\frac{\alpha}{2}} \downarrow \)

Recall that width = \( 2 \left( Z_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \right) \)

So as \( \alpha \uparrow \rightarrow \text{level of confidence} \downarrow \Rightarrow Z_{1-\frac{\alpha}{2}} \downarrow \rightarrow \text{narrower confidence interval} \)
\( \uparrow \text{greater precision} \)

and when \( \alpha \downarrow \rightarrow \text{level of confidence} \uparrow \Rightarrow Z_{1-\frac{\alpha}{2}} \uparrow \rightarrow \text{wider CI} \)
\( \downarrow \text{smaller precision} \)

**Example:** For the female height example

\( \alpha = 0.05 \Rightarrow 95\% \text{ CI for } \mu \Rightarrow [163.3, 165.5] \)

\( \alpha = 0.01 \Rightarrow 99\% \text{ CI for } \mu \Rightarrow [160.6, 160.2] \)

\( \pm \) which interval is wider? Which interval is more precise?

In most projects, people calculate 95\% CI \( \pm \) This is the most common case.

The table below gives you values of \( Z_{1-\frac{\alpha}{2}} \) for other confidence levels

<table>
<thead>
<tr>
<th>( \frac{\alpha}{2} )</th>
<th>0.15</th>
<th>0.1</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.85</td>
<td>0.90</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>1.96</td>
<td>1.44</td>
<td>1.645</td>
<td>1.96</td>
<td>2.58</td>
</tr>
</tbody>
</table>

\( \pm \) check this from the \( Z \)-table.
Case II: Confidence Interval for $\mu$ when $n$ is small ($n < 30$)

If $n$ is small ($n < 30$), a $(1-\alpha) \times 100\%$ CI for $\mu$ is

$$
\bar{x} \pm t_{(1-\alpha/2), n-1} \times \frac{s}{\sqrt{n}}
$$

where $t_{(1-\alpha/2), n-1}$ is the value of a $t$-random variable to the right of which the area is $\frac{\alpha}{2}$. It is obtained from Table C.5, page 332.

When $n < 30$, this CI is valid under the following assumption:

"The parent popn is approximately normally distributed."

Note: The interpretation of CI discussed for case I also applies here.

The new things in case II are: there is an assumption for appropriateness of the CI and you need to learn about the $t$-random variable and how to use Table C.5.

The $t$-distribution

is a continuous probability distribution. If we take a large number of random samples of the same size, $n$, from a normal distn with known mean, $\mu$, then the probability distn of the statistic

$$
t = \frac{\bar{x} - \mu}{s / \sqrt{n}}
$$

follows a $t$ distn with $n-1$ degrees of freedom.

The $t$ distn is symmetrical and unimodal.

- for different values of $n$, different $t$-dists will be obtained.
- for large $n$, the $t$-distn approaches the standard normal distn.
- for small $n$, the $t$-distn is flatter and has higher tails than the standard normal (see fig. 9.6, page 124).
How to use the t-table [Table 5.3]

Find the values of the t-random variable, for the following cases

1. $t_{0.05, 10}$
2. $t_{0.05, 14}$
3. $t_{0.01, 25}$
4. $t_{0.10, 30}$

Note: The table gives areas to the right.

$t_{0.05, 10} = 1.812$
$t_{0.05, 14} = 1.761$
$t_{0.01, 25} = 2.485$
$t_{0.10, 30} = 1.310$

Read about the general meaning of degrees of freedom ($v$)
Section 9.7.1, page 125

Example: CI for $\mu$ when $n \leq 30$

To find a 95% CI for mean height of female sunflower students, heights of a random sample of 9 females were obtained. The sample mean and SD were $\bar{x} = 162.3$, $s = 5.3$. Find 95% CI for $\mu$.

Since $n$ is less than 30, use $\frac{t_{v}}{2}$.

$\bar{x} = 162.3$
$s = 5.3$
$n = 9$

$\alpha = 0.05 \Rightarrow t_{\frac{\alpha}{2}, n-1} = t_{0.025, 8} = 2.306$

$\bar{x} \pm t_{\frac{\alpha}{2}, n-1} \times \frac{s}{\sqrt{n}} \Rightarrow 162.3 \pm 2.306 \times \frac{5.3}{\sqrt{9}} \Rightarrow [158.23, 166.37]$

We are 95% confident that the true value of $\mu$ lies in this interval. The interval is valid.
Confidence Interval for

Estimating Sample Size When Estimating
The Mean of a Population (Section 9.6)

We want to estimate \( \mu \) of a population. The question is how much observations (n) should we collect? It depends on:

- The precision with which \( \mu \) is to be estimated.
- The variability of the measurements (\( \sigma \)).

You need a priori estimate of \( \sigma \) - either from:

- A pilot experiment.
- Your own knowledge or experience, or
- Based on previous research results.

**Example**: We want to find a 95% CI for \( \mu \). We specify the error term to be 1 and from experience we guess that \( \sigma \) to be around 10.

So, how many subjects or observations should we sample? \( n \)?

**Solution**

\[
95\% \text{ CI } \Rightarrow \alpha = 0.05 \\
\text{error term } = \frac{t_{0.025, n-1} \times \hat{\sigma}}{\sqrt{n}} = t_{0.025, n-1} \times \frac{10}{\sqrt{n}} = 1
\]

Since we do not know \( n \), we cannot find \( t_{0.025, n-1} \), hence one solution is to always assume that \( n \) is large, if you select the values of \( t_{0.025, n-1} \) from table C.5, it is around 2 for large \( n \).

So, \( t_{0.025, n-1} \approx 2 \) for large \( n \).

\[
\frac{2 \times 10}{\sqrt{n}} = 2 \Rightarrow \sqrt{n} = 20 \\
\Rightarrow n = 400
\]
Confidence Interval for a binomial proportion \( p \)

\[ p \] proportion of success in the popn

we want to find a CI for \( p \)

Take a random sample of Size \( n \), let \( x \) be the number of successes in the sample, then a \((1-\alpha) \times 100\%\) CI for \( p \) is

\[
\frac{x}{n} \pm Z_{1-\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}
\]

where \( \hat{p} = \frac{x}{n} \)

The sample proportion,

Assumptions:
1) \( x > 5 \)
2) \( n-x > 5 \)

Example

let \( p = \) propn of voters who will vote for A

find a 95% CI for \( p \)

\( \alpha = 0.05 \Rightarrow Z_{1-\alpha} = Z_{0.975} = 1.96 \)

\( \hat{p} = \frac{x}{n} = \frac{110}{200} = 0.55 \)

95% CI for \( p \) is

\[
\frac{x}{n} \pm Z_{1-\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \Rightarrow 0.55 \pm 1.96 \times \sqrt{\frac{0.55(1-0.55)}{200}} \\
\Rightarrow 0.55 \pm 1.96 \times 0.035778 \\
\Rightarrow 0.55 \pm 0.068949 \\
\Rightarrow [0.481, 0.619]
\]

We are 95% confident that the true propn of voters in the popn who will vote for candidate A are between 48 and 62 percent.

Answer the following:
1) What is the point estimate \( \hat{p} \)?
2) Is 0.58 a plausible value of \( p \)?
3) Are the assumptions met?
4) What is the width of the CI?
Choice of Sample Size When Estimating a Binomial Probability (Section 7.9).

We want to estimate the proportion of success (p) in a popn? How much subjects shall we survey? N?

Example

We want a 95% CI for p. We wish to estimate p to "within an error term of 0.02" find N.

\[
\alpha = 0.05 \Rightarrow Z_{1-\alpha/2} = Z_{0.975} = 1.96
\]

error term = \[
Z_{1-\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{N}} = 0.02
\]

From previous survey, or based on your knowledge, you need to provide some value for \(\hat{p}\).

Let us use \(\hat{p} = 0.55\)

\[
1.96 \times \sqrt{\frac{0.55(1-0.55)}{N}} = 0.02
\]

\[
\sqrt{\frac{0.2475}{N}} = \frac{0.02}{1.96}
\]

\[
0.2475 \div \frac{1}{N} = \left[0.010204\right]^2
\]

\[
N = \frac{0.2475}{0.00104}
\]

\[
= 237.699 \approx 237
\]

So we need to interview about 237 people to estimate \(p\) with an error term of 0.02.

New material:
The 2nd section of chapter 9 will follow.
Confidence Intervals for two populations:
comparing means

There are two cases when comparing means of two populations:
I. Case where the two populations are independent
II. Case where the two popns are dependent (paired popns)

Case I: Confidence Interval for the difference in the means of
two independent populations

Example: Suppose we want to compare the mean age of female vs.
freshman WVU students. To accomplish this, suppose we take
a random sample of \( n_1 \) female freshman students and \( n_2 \) male
freshman students.

\[
\begin{align*}
\mu_1 &= \text{The mean age of all female freshman students} \\
\sigma_1 &= \text{The SD of ages of all} \\
\overline{x}_1 &= \text{The sample mean of their ages} \\
S_1 &= \text{The sample SD of their ages}
\end{align*}
\]

\[
\begin{align*}
\mu_2 &= \text{The mean age of all freshman male} \\
\sigma_2 &= \text{The SD of ages of all} \\
\overline{x}_2 &= \text{The sample mean of their ages} \\
S_2 &= \text{The sample SD of their ages}
\end{align*}
\]

Always remember that we will never know the values of \( \mu_1 \) and \( \mu_2 \) ⇒ they are pop parameters.
So our objective is based on the information from the two samples we want to
compare \( \mu_1 \) and \( \mu_2 \). Is \( \mu_1 = \mu_2 \) or \( \mu_1 - \mu_2 = 0 \) ⇒ we want to answer this:

Data Structure

<table>
<thead>
<tr>
<th>Females (sample1)</th>
<th>Males (sample2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>subject</td>
<td>(age)</td>
</tr>
<tr>
<td>x1</td>
<td>x1</td>
</tr>
<tr>
<td>x2</td>
<td>x2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>x1</td>
<td>x1</td>
</tr>
</tbody>
</table>

The subjects in sample(1) and sample(2) are not the same or are not paired. They
come from two different populations. Hence the samples are independent.
Independent of variables: Two variables are independent if knowledge of the value of one variable provides no information about the value of another variable. For example, knowing the age of the first female in sample 1, does not give you any clue about the age of the first male in sample 2. \( \Rightarrow \) So the two samples are independent.

A \((1 - \alpha) \times 100\%\) CI for \(\mu_1 - \mu_2\) is

\[
\bar{x}_1 - \bar{x}_2 \pm t \left( \frac{\alpha}{2}, n_1 + n_2 - 2 \right) \times s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}
\]

Where

\[
s = \sqrt{\frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}}
\]

is the pooled standard deviation

Assumptions: To use the above formula, the following assumptions should be satisfied:

1) The measurements in each popn must be approximately normal (this assumption is less critical if \(n_1\) and \(n_2\) are large)

2) The population variances (or standard deviations) must be equal; that is, \(\sigma_1^2 = \sigma_2^2\). This assumption is called Homoscedasticity of variance.

Example: There are two kinds of calculator batteries produced by a company, \(A = E\)verst, \(B = J\)ordovac

We want to compare the mean life time of these two batteries? Compare \(\mu_1\) vs. \(\mu_2\) ?

**Sample of 45 E\textit{verst} batteries**

- \(n_1 = 45\)
- \(\bar{x}_1 = 125.245\) hours
- \(s_1 = 34.890\) hours

**Sample of 50 J\textit{ordovac} batteries**

- \(n_2 = 50\)
- \(\bar{x}_2 = 120.051\) hours
- \(s_2 = 42.800\) hours
Check assumptions:
1) $n_1 = 30$, $n_2 = 30$, the assumption of normality is not very important.
2) Homoscedasticity of variance: $S_1 = 34.890$, $S_2 = 42.890$
   $S_{\text{largest}}^2 \approx S_{\text{smallest}}^2 = 42.890^2 / 34.890^2 \approx 1.5$, if this ratio was more than 4, we will be concerned about the assumption of equal variance.

S. both assumptions are fairly satisfied.

Find a 95% CI for $\mu_1 - \mu_2$

\[
(\bar{x}_1 - \bar{x}_2) \pm t_{0.025} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}
\]

\[
S = \sqrt{\frac{S_1^2 (n_1 - 1) + S_2^2 (n_2 - 1)}{n_1 + n_2 - 2}} = 39.257
\]

\[
\begin{align*}
&= \sqrt{\frac{34.890^2 (45) + 42.890^2 (50)}{45 + 50 - 2}}
&= 39.257
\]

\[
\frac{185.245 - 120.05}{39.257} \pm 1.980 \times 39.257 \times \sqrt{\frac{1}{45} + \frac{1}{50}}
\]

\[
5.95 \pm 1.980 \times 39.257 \times \sqrt{\frac{1}{45} + \frac{1}{50}}
\]

\[
5.95 \pm 15.972
\]

[10.77, 21.17]

Conclusion: We are 95% confident that the mean difference in lifetime between the two batteries lies in the interval $[-10.77, 21.17]$.

** Since this interval contains zero, we are 95% confident that there is no difference between the two batteries with respect to lifetime.

Please do more exercises from the end of the chapter.

In the above example:
- What is the point estimate of $\mu_1 - \mu_2$?
- What is the margin of error for the point estimate of $\mu_1 - \mu_2$?
- What is the width of the CI?
- Find the 99% CI for $\mu_1 - \mu_2$ and interpret your result.
- Based on the 95% CI, which battery do you recommend using to buy, and why?
Case II: Confidence Interval for the mean \( \bar{d} \) of a population of differences: paired samples

Example:
Suppose you want to determine if a pill has the undesirable effect of reducing blood pressure of the user. There are 15 women available for this study.

1. You record the initial blood pressure of these 15 women ⇒ call this measurements \( X \).
2. Six months after they use the pill, you record their blood pressure again. Call this measurements \( Y \).

Question: Does the pill causes reduction of blood pressure?

Your data looks as follows:

<table>
<thead>
<tr>
<th>Subject</th>
<th>Initial Blood Pressure</th>
<th>Blood Pressure After Taking the Pill</th>
<th>( d = (X - Y) )</th>
<th>Difference in Blood Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>68</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>72</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>72</td>
<td>62</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>15</td>
<td>84</td>
<td>74</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Note: Each set of measurements are obtained from the same subject (woman). There are two measurements on the same woman, one before the treatment and one after the treatment. This makes the two sets of measurements dependent. Knowing the initial blood pressure of a woman gives you a clue about her blood pressure six months after. So \( X \) and \( Y \) are dependent variables.

In this example, it seems like there are two populations, but we are interested only in the difference: \( d \) in blood pressures. The 15 observations (differences) in the fourth column (labeled \( d \)) are like a sample from one population, namely the population of differences \( d \) in blood pressure before and after woman use the pill.

Note: The differences were calculated as: \( d = \) before - after, hence a high positive value of \( d \) indicates that the pill is reducing blood pressure.
Now, consider this as a sample of size \( n = 15 \) from a population of differences.

**Notation**

- \( \mu_d \): The mean of the population of differences
- \( \bar{d} \): The mean of the sample of differences
  \[ \bar{d} = \frac{\Sigma d}{n} \]
- \( s_d \): The standard deviation of the sample differences
  \[ s_d = \sqrt{\frac{\Sigma d^2 - (\Sigma d)^2}{n-1}} \]

Find a 95% CI for \( \mu_d \).

A \((1-\alpha) \times 100\%\) CI for \( \mu_d \) is

\[ \bar{d} \pm t(\frac{\alpha}{2}, n-1) \times \frac{s_d}{\sqrt{n}} \]

**Assumption:** To use the formula, we assume that the differences are normally distributed. If \( n \) is large (\( n > 30 \)), this assumption is not critical.

For the blood pressure example:

Suppose \( \bar{d} = \frac{\Sigma d}{15} = 8.80 \)

\( s_d = 10.98 \)

\( n = 15 \)

95% CI for \( \mu_d \):

\[ 8.80 \pm 2.145 \times \frac{10.98}{\sqrt{15}} \]

\[ 8.80 \pm 2.145 \]

\[ [6.74, 10.86] \]
Conclusion: We are 95% confident that the mean difference (reduction) in blood pressure is between 2.72 and 14.88.

Since the interval does not include zero, we are 95% confident that the pill indeed reduces blood pressure.

Important: There is a very good example on page 128, about two methods of teaching children to read. Please read this example.

End of Chapter 9

Summary

Point estimate, their disadvantage
Confidence interval estimate
Meaning of a 95% confidence interval
Why a width of a confidence interval is important
Factors affecting the width of a CI
Calculating N when estimating μ and p.

CI for μ

If n ≥ 30

\[ \bar{x} \pm z_{1-\alpha/2} \times \frac{\sigma}{\sqrt{n}}. \] (No assumption)

If n < 30

\[ \bar{x} \pm t_{\alpha/2, n-1} \times \frac{S}{\sqrt{n}}. \] (Need an assumption, what is it?)

Remember: If n is large, formulas (a) and (b) yield about the same result, so you can use either.

CI for p

\[ \hat{p} \pm z_{1-\alpha/2} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}. \] (Under what assumption)

CI for μ₁ - μ₂

\[ \bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2, n_1+n_2-2} \times S \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}. \] (Under what assumptions)

\[ S = \sqrt{\frac{s_1^2(n_1-1) + s_2^2(n_2-1)}{n_1+n_2-2}}. \]

CI for d

\[ \bar{d} \pm t_{\alpha/2, n-1} \times \frac{S_d}{\sqrt{n}}. \]

Focus on interpretation of all these confidence intervals, please at least try the exercises at the end of the chapter.